Newton found that the gravitational force of attraction $|F_g|$ between two masses is:

- directly proportional to the product of the two masses, so $|F_g| \propto (m_1)(m_2)$
- inversely proportional to the square of the distance between their centers, so $|F_g| \propto 1/r^2$
Combining these two relationships, we get:

\[ |F_g| \propto \frac{m_1 m_2}{r^2} \]

The proportional sign (\(\propto\)) doesn’t mean “equal,” so this is not an actual equation.

If a constant is added, we have an equation:

\[ F_g = \frac{G m_1 m_2}{r^2} \]

where:

- \(F_g\) = gravitational force (N or kg\(\cdot\)m/s²)
- \(m_1, m_2\) = object masses (kg)
- \(G\) = universal gravitational constant (6.67 x 10⁻¹¹ N\(\cdot\)m²/kg²)
- \(r\) = center-to-center distance (m)
Newton’s Law of Universal Gravitation

✧ Cavendish used a torsion balance to find G:

\[ G = \frac{|F_g| r^2}{m_1 m_2} \]

✧ The unit for G is N·m²/kg² from this equation

✧ He measured \( F_g \) between two known masses (\( m_1 \) and \( m_2 \)) separated by distance \( r \)

✧ The constant, \( G \), could then be calculated using \( F_g \), \( m_1 \), \( m_2 \) and \( r \)

Gravitational Field Strength

✧ The following is an equation for gravitational field strength:

\[ g = \frac{Gm_1}{r^2} \]

**where:**

- \( g \) = acceleration due to gravity (m/s²)
- \( G = 6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2 \)
- \( m_1 \) = mass of gravitational source (kg)
- \( r \) = center-to-center distance from gravitational source to object (m)
This shows us acceleration due to gravity only depends on the mass of the gravitational source, not the object affected!

A field is a region of influence that surrounds an object.

A field exists as long as there is a source.

A force is the effect of an external field on an object that enters it.

A gravitational source is surrounded by a gravitational field, because it has mass!

The strength of the gravitational field is given by “g” (acceleration due to gravity).

The gravitational field is:

- strong closer to the source
- weak farther from the source
There are two ways to find gravitational field strength (g) in units of N/kg or m/s²:

1) If an object is on the surface and weight is known, \( F_g = mg \) may be used.

2) If an object is on the surface or in orbit, the following equation may be used:

\[
g = \frac{Gm_1}{r^2}
\]

Einstein (1915)’s General Theory of Relativity:
- space (3D) and time (1D) are united (4D)
- mass warps space-time
- gravity is simply the effect of mass warping space-time

In Einstein’s gravity field, objects and light follow a curved path around the source.
Uniform Circular Motion

- Properties of uniform circular motion:
  1) object is pulled toward the center of the circle by the “centripetal force”
  2) acceleration is directed to the center, called “centripetal acceleration”
  3) speed of the object is constant
  4) velocity of the object keeps changing, because direction keeps changing

Centripetal force: object in circular motion exerts an action force towards the center

Centrifugal force: a reaction force of equal magnitude is exerted towards the object

Newton’s 2nd law may be applied to centripetal acceleration:

\[ F_c = m\ddot{a}_c \]
Period and frequency may be related for circular motion:

\[ T = \frac{1}{f} \]

OR

\[ f = \frac{1}{T} \]

**where:**

- \( T \) = period or time for a complete revolution (s)
- \( f \) = frequency or number of complete revolutions per second (rps or Hz)

To convert rpm to rps, divide by 60!

The following equation describes distance and time of one revolution in a circle:

\[ v = \frac{2\pi r}{T} \]

**where:**

- \( v \) = tangential speed (m/s)
- \( r \) = radius (m)
- \( T \) = period (s)
For uniform circular motion, centripetal force is related to mass, speed and radius by:

$$F_c = \frac{mv^2}{r}$$

where:

- $F_c$ = centripetal force (N)
- $m$ = mass (kg)
- $r$ = radius (m)
- $v$ = speed (m/s)

For uniform circular motion, centripetal force is related to mass, radius and period by:

$$F_c = \frac{4\pi^2mr}{T^2}$$

where:

- $F_c$ = centripetal force (N)
- $m$ = mass (kg)
- $r$ = radius (m)
- $T$ = period (s)
Converting period into frequency gives us another equation for centripetal force:

\[ F_c = 4\pi^2 mrf^2 \]

**where:**
- \( F_c \) = centripetal force (N)
- \( m \) = mass (kg)
- \( r \) = radius (m)
- \( f \) = frequency (rps or Hz)

Combining a centripetal force equation with Newton’s 2nd law:

\[ a_c = \frac{v^2}{r} \]

**where:**
- \( a_c \) = centripetal acceleration (m/s²)
- \( v \) = tangential speed (m/s)
- \( r \) = radius (m)
Centripetal Force and Acceleration

Two other equations for centripetal acceleration:

\[ a_c = \frac{4\pi^2 r}{T^2} \]
\[ a_c = 4\pi^2 rf^2 \]

where:

- \( a_c \) = centripetal acceleration \( (m/s^2) \)
- \( r \) = radius \( (m) \)
- \( T \) = period \( (s) \)
- \( f \) = frequency \( (\text{rps or Hz}) \)

Vertical Circular Motion

Three equations are arranged into a net force equation for vertical circular motion:

\[ \vec{F}_{c(\text{net})} = \vec{F}_T + \vec{F}_g \]

- The magnitudes of \( \vec{F}_c \) and \( \vec{F}_g \) are constant for the object
- The magnitude of \( \vec{F}_T \) depends on location
When the object is at the bottom (highest tension):
- $\vec{F}_c$ is up (+)
- $\vec{F}_T$ is up (+)
- $\vec{F}_g$ is down (-)
- Direction (and sign) of $F_c$ must be assigned by you!

When the object is at the top (lowest tension):
- $\vec{F}_c$ is down (-)
- $\vec{F}_T$ is down (-)
- $\vec{F}_g$ is down (-)
- Again, direction and sign of $F_c$ are assigned by you.
Vertical Circular Motion

- It is also possible to have vertical motion with zero tension force on the object:
  \[ F_{c nett} = F_T + F_g = 0 N + F_g \]
- When this happens, \( |F_c| = |F_g| \) (scalar calc.)
- A bare minimum of centripetal force is being supplied to overcome gravitational force
- This may occur at the top of vertical circular motion, with \( v \) as the minimum speed needed

Horizontal Circular Motion

- When there is surface friction, the \( F_f = \mu F_N \) equation must be used with centripetal force
- For a flat surface:
  \[ F_N = |F_g| = |mg| \]
- Therefore, the magnitude of friction force is:
  \[ F_f = \mu F_N = \mu mg \]
If an object is moving in a horizontal circle, friction and centripetal force are balanced:
\[ F_f = F_c \]

- We can substitute in and eliminate mass:
\[ \mu mg = \frac{mv^2}{r} \]

- This results in the following equation:
\[ \mu g = \frac{v^2}{r} \]

We can manipulate this equation for the minimum speed to maintain a turn:
\[ v = \sqrt{\mu gr} \]

**where:**
- \( v \) = minimum speed (m/s)
- \( \mu \) = coefficient of friction
- \( g \) = acceleration due to gravity (m/s\(^2\))
- \( r \) = radius (m)

**MEMORIZE!**
Kepler’s Laws

Kepler’s 3 laws:

1) The planets move about the sun in elliptical paths, with the sun at one focus of the ellipse (slight exaggeration):

http://www.surendranath.org/Applets/Dynamics/Kepler/Kepler3Applet.html

2) A straight line joining the sun and a given planet sweeps out equal areas in equal times:

http://www.surendranath.org/Applets/Dynamics/Kepler/Kepler1Applet.html
Kepler’s 3 laws:

3) The square of the period of revolution of a planet about the sun is directly proportional to the cube of its mean distance from the sun:

\[ T^2 \propto r^3 \]

With the addition of a proportionality constant \( K \), we get an equation:

\[ T^2 = Kr^3 \]

Kepler’s constant applies to any planetary orbit, so two planets may be compared:

\[
K = \frac{T_a^2}{r_a^3} = \frac{T_b^2}{r_b^3}
\]

where:

- \( K = \) Kepler’s proportionality constant \( (2.975 \times 10^{-19} \text{ s}^2/\text{m}^3) \)
- \( T_a, T_b = \) orbital periods (s)
- \( r_a, r_b = \) mean orbital radii (m)

\[ a = 1^{\text{st}} \text{ planet} \]
\[ b = 2^{\text{nd}} \text{ planet} \]
The International Astronomical Union (2006) defined a planet as a celestial body that:

1) has sufficient mass to have a “nearly round shape”
2) is in orbit around the Sun, or other star
3) has "cleared the neighbourhood" around its orbit

For an orbiting object, gravitational force is the same magnitude as centripetal force.

We can balance the magnitudes of centripetal force with gravitational force:

\[ |F_c| = |F_g| \]

Using this, we can derive an equation for satellite speed or escape velocity:
Planetary and Satellite Motion

\[ v = \sqrt{\frac{G m_p}{r}} \]

where:
- \( v \) = satellite speed/escape velocity (m/s)
- \( G \) = Universal Gravitational Constant 
  \[ (6.67 \times 10^{-11} \text{ N}\cdot\text{m}^2) \]
- \( m_p \) = mass of the planetary object (kg)
- \( r \) = center-to-center distance between the planetary object and satellite (m)

REMINDER: if the orbital radius is given, it is “r”

Planetary and Satellite Motion

- It is not possible to measure the mass of a new celestial body directly
- We collect data on how the celestial body affects a probe (which is a satellite)
- Orbital data for the satellite helps us to measure the mass of the new celestial body
The previous equation may be manipulated to find mass of the planetary object:

\[ m_p = \frac{v^2 r}{G} \]

- \( m_p \) = mass of the celestial body (kg)
- \( v \) = speed of the satellite (m/s)
- \( r \) = orbital radius of the satellite (m)
- \( G = 6.67 \times 10^{-11} \)

Two other equations may be derived for \( m_p \):

Using another equation for centripetal force, involving period, we get:

\[ m_p = \frac{4\pi^2 r^3}{GT^2} \]

- \( m_p \) = mass of the celestial body (kg)
- \( r \) = orbital radius of the satellite (m)
- \( G = 6.67 \times 10^{-11} \)
- \( T \) = orbital period of the satellite (s)
Using another equation for centripetal force, involving frequency, we get:

\[ m_p = \frac{4\pi^2 r^3 f^2}{G} \]

- \( m_p \) = mass of the celestial body (kg)
- \( r \) = orbital radius of the satellite (m)
- \( G = 6.67 \times 10^{-11} \)
- \( f = \) frequency of orbits (Hz or s\(^{-1}\))

Remember how to derive both equations for mass (or memorize them)

For the following equations, a graph may show us the relationship between variables:

\[ T^2 = Kr^3 \quad v = \frac{2\pi r}{T} \quad T = \frac{1}{f} \]

\[ v^2 = \frac{G m_p}{r} \quad m_p = \frac{4\pi^2 r^3}{GT^2} \quad m_p = \frac{4\pi^2 r^3 f^2}{G} \]

Important variables here are mass, speed, frequency and radius
The two main types of mathematical relationships that are important here:

1. Variables are “directly proportional”:

\[ y \propto x \]

2. Variables are “inversely proportional”:

\[ y \propto \frac{1}{x} \]